A Revealed Preferences Method for Evaluating Redistricting Intent

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Abstract

We place evaluating intent as a motive behind a redistricting plan into a formal quantitative micro-economic framework to evaluate existing and emerging methods, and find that these methods are statistically flawed. In place of classical statistical tests, we formalize a method of revealed preferences to probe intent by comparing aspects of plans that were feasible, but *not* selected. This method has been used in an informal, ad-hoc, manner in redistricting cases, but is not well documented and has never been rigorously analyzed. Our method has four advantages. First, it is easily interpretable. Second, it can be applied using only the data available to the original planners and does not require estimating the outcomes of hypothetical elections. Third, lacking sophisticated optimization technology, the basic method can be applied using hand drawn maps. Finally, it is the only quantitative method for determining intent, so far proposed, that is statistically sound.

Discriminatory intent and results are intimately intertwined in redistricting law and academic research. While assessment of effects through quantifiable methods is possible, incomplete, ambiguous or misleading legislative statements and actions complicate assessment of intent. Thus, a legal standard based on legislative intent is typically more difficult to satisfy than a standard based on the reasonably foreseeable effects of legislation. This point has not been lost on the courts and lawmakers: When the Supreme Court articulated an intent-oriented standard of "purposeful discrimination" for racial gerrymandering in *Mobile* v *Bolden* 446 US 55 (1980), Congress amended the Voting Rights Act in 1982, resulting in the adoption of the three-prong results test in *Thornburg* v *Gingles* 478 U.S. 30 (1986). The invention and refinement of statistical methods to probe discriminatory results of a redistricting plan followed.

In the 1990's intent was re-emphasized in the lexicon of redistricting law. In *Miller v. Johnson*, 515 U.S. 900 (1995), the Supreme Court articulated a "predominant factor" standard in racial gerrymandering, finding that race must predominate above all other interests, such as partisan and incumbent interests, for a districting plan to be held up to strict scrutiny.[1] Ad-hoc methods were again invented and refined to probe if race was the predominant factor in the results of a districting plan.

In this paper, we accept, *arguendo*, that predominant racial intent is harmful, and hence that the question of how to determine institutional intent (predominant and otherwise) is both meaningful (at least in some situations) and interesting. We follow Kousser (1991) in arguing that institutional intent is best understood conceptually in terms of the actions of counterfactual legislatures, and how this concept of intent is directly translatable into statistical terms, (the 'Bayes Factor') following Altman 2002.

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[1] Applying this test, the Court struck down the Eleventh congressional district in Georgia, one of only two districts (out of eleven) that had a majority of black voters, in a state with a population that was more than one quarter black.

We find however, that statistical methods are likely to be impossible to implement, because taking the requisite random samples from the set of possible districts conditioned on particular intents is probably computationally intractable. In fact, all methods to determine intent in redistricting that are in current are unreliable, statistically biased, or ignore evidence of relevant competing explanations of intent. Even the best methods examine criteria that are merely correlated with gerrymandering, without capturing the constraints that a human cartographer must face.

Since statistical models to evaluate intent directly are likely to be computationally impossible (although it is statistically coherent), we develop an alternative inferential strategy (WARP) that is still consistent with the same 'counterfactual' concept of intent. In place of classical statistical tests and their variants, we formalize a method of revealed preferences to probe intent by comparing aspects of plans that were feasible, but *not* selected. This method has been used in an informal, ad-hoc, manner in redistricting cases, but is not well documented and has never been rigorously analyzed. Our method has five advantages. First, it is easily interpretable. Second, it can be applied using only the data available to the original planners and does not require estimating the outcomes of hypothetical elections. Third, lacking sophisticated optimization technology, the basic method can be applied using hand drawn maps. Fourth, it is more consistent with the knowledge that distracters are know to have than statistical methods – unlike the Bayes factor and other method, it does not implicitly assume that the districter was aware of all possible plans. Finally, it is the only quantitative method for determining intent, so far proposed, that is statistically sound.

Section 2. The Supreme Court Reveals a Predominant Intent Standard

The Supreme Court has walked a winding path between intent and effect in case law for more than one hundred and thirty years. As Kousser (1991) notes, "Moving from objective intent and effect in *Yick Wo,* to effect alone in *Williams,* to political questions in *Giles,* to objective intent and affect with an emphasis on the forever in *Guinn*, the Supreme Court followed a confusing and contradictory path on occupational and voting rights" (1991: 690). While in later years the judicial path "…was full of sharp curves and repeated switchbacks, and judges careened from effect to intent in efforts to avoid what they perceived as various slippery slopes on both sides…" (1991: 694).

What is collective intent? Is there a coherent notion of collective intent beyond simply foreseeable effects? Suppose that a redistricting plan is adopted by a majority of a legislature. Also suppose, of that voting majority, more than half of the legislators vote for the plan for purely non-racial reasons. Race is the top priority for a small coalition of legislators, however, and without their votes the plan could not have passed. Is the districting plan racially motivated?

This is a difficult question to answer because, at a deep level, the notion of *collective* legislative intent is ill-posed. Arrow (1951) shows that individual rationality, and especially the concept of individual preference, cannot be consistently attributed to collective bodies that use majority voting rules. Collective intent, which seems to rely on the notion of collective preference, must fail to be meaningful in some cases of supposed racial gerrymandering (Kousser 1991). Justice Scalia (1997), too, argues that a legislature can rarely, if ever, said to have had any recognizable collective intent. Even if the motives of individual legislators are sufficiently unified that collective intent can be said to exist, or that we simply choose to ignore the collective preference problem, quantifying the *predominant* factor of this collective intent remains fraught with difficulty.[2] As Ely (1998) writes, "Drawing a voting district involves an infinity of choice."

Even if we could assign collective intent to individual behavior, statements of intent cannot be accepted at face value. On one hand, legislators make exaggerated statements to court constituents. On the other, legislators advocate *prima facie* neutral reasons to achieve politically motivated goals.[3] Consider the 1973 racial gerrymander of Mississippi county supervisors' districts chronicled by Parker (1990: 155-66). In response to *Allen* v *State Board of Elections* 393 U.S. 544 (1969), which invalidated

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2 Prior to *Miller* the Court established competing standards for interpreting intent, as discussed in Kousser (1991). In *Feeney*, the Supreme Court interpreted a racial motive as action that is taken in part because of race, while in *Arlington Heights*, racial intent occurred where action would not have been taken except for race.

at-large elections for Mississippi's county supervisor districts, Hinds County officials drew districts based on the goal of equalizing population, land area, county road mileage and bridges (road maintenance was one of the responsibilities of county supervisors). Although these goals were facially neutral, the effect of their application was not. The simultaneous equalization of road mileage and population led to districts that split urban areas, where black voters tended to be concentrated, and thus resulted in a formula for a racial gerrymander.

At present there is significant uncertainty over how to recognize "predominant intent," especially in the context of districting plans that may represent a compromise between competing goals such as protection of incumbents, partisan advantage, and idiosyncratic desires such as including important landmarks into a district. Justice Stevens's dissent in *Abrams* v *Johnson* 521 U.S. 74 (1997), laments the continued lack of guidance on this question (Canon 1999: 82). The courts continue to withhold judgment on these methodologies, and there is no case law that establishes statistical tests for predominant intent. Nor is there a generally accepted methodology within the social sciences for making such determinations, although specific historical arguments made within the context of a particular case can be, on occasion, compelling.[4]

Nevertheless, the Court continues its search for a standard in gerrymandering. From *Bolden*, to *Gingles*, and more recently *Miller*, the Court has struggled with interpreting the 14th Amendment and the Voting Rights Act as they apply to racial gerrymandering. Even as the Court pronounced the predominant factor standard in Miller, Justice O'Conner, writing in a concurrence (as well as joining the majority) articulated a seemingly different standard based on traditional districting principles: "To invoke strict scrutiny, a plaintiff must show that the State has relied on race in *substantial disregard of customary*

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3 Nozick (1993) uses the term "second-level bias" to refer to this selection of standards or procedures, which will be applied evenly, but which are chosen to advantage a particular group.

4 Nor is the Court's determination of the facts in these racial gerrymandering cases uncontroversial. For example, many scholars argue that the infamous *Shaw* districts in North Carolina, considered to be an egregious racial gerrymander, were actually heavily motivated by incumbent and political interests (Canon 1999; Gronke and Wilson 1999)

and traditional districting practices."[5] The Court is more skeptical of the applicability of intent or effects standards in partisan gerrymandering. Justice Scalia, writing for the plurality in *Vieth*, claims that it not possible for the courts to determine partisan effect or intent in redistricting cases, and thus political gerrymandering cases should be deemed nonjusticiable.[6]

Section 3. Previous Attempts to Quantify Intent

Scalia's opinion has not stopped social scientists from pursuing quantitative methods to evaluate intent. Many of these are ad-hoc methodologies, based neither on a full statistical model of the redistricting process or on published research. For example, *In Re 2001 Redistricting Cases* (Case No. S-10504), Dr. Viscussi, Professor of Law and Economics at Harvard Law School, applied a standard *t*-test to determine if the population deviations of Alaska's state legislative districts were intentionally imbalanced. In *Hunt v. Cromartie*, the court refers to Dr. David Peterson analysis, using the demographics of the census regions on the boundary of the challenged district: "He concluded that the State included the more heavily Democratic precinct much more often than the more heavily black precinct, and therefore, that the data as a whole supported a political explanation at least as well as, and somewhat better than, a racial explanation." We have little doubt that other ad hoc methods have been applied elsewhere, and are more often buried in depositions than documented in journals -- our experience in litigation and discussion with others involved in redistricting suggests that ad hoc analyses are far from rare.

A more formal and well-studied methodology exists to evaluate intent by measuring effects of redistricting plans. Since 1970, geographers and a few political scientists have developed and advocated automated redistricting techniques to evaluate plans by contrasting them with the set of alternative plans

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5 This alternative principle is also stated her earlier plurality opinion in *Shaw v. Reno* 509 U.S. 630 (1993), and its later plurality opinion in *Bush* v. *Vera* 517 U.S. 952 (1996).

6 Many social scientists would beg to differ with Scalia's first claim. In fact, well-known and well-tested statistical methods exist for evaluating the partisan effects of redistricting (see Gelman and King 1994; Kousser 1996). However, methods for evaluating indirect evidence of intent, beyond showing evidence of "reasonably foreseeable" partisan effect, are much less well developed and understood.

that could have been created (Shepherd and Jenkins 1970; Gudgin and Taylor 1979; Rossiter and Johnston 1984). The most recent and sophisticated version of this method is offered by Cirincione, Darling and O'Rourke (2000, henceforth "CDR"). Some of the particular features of the CDR approach and the claims made by them are unique, so it is instructive to examine this technique's context.

CDR propose a seemingly simple and straightforward method for determining legislative intent in racial gerrymandering cases.[7] These scholars proceed in three steps. First, they construct computer "algorithms" to generate districting plans on the basis of "neutral" criteria: using only contiguity, equal population, preservation of counties, and compactness. Second, they use the algorithms to generate a large number of "randomly sampled" plans. Third, they compare the plan in question to the plans generated by their method. They conclude the redistricting authority intended a racial gerrymander if few of the generated plans have at least as many minority-majority districts as the plan in question. The method does not rely on foreseeable effect alone. The mere presence of an effect (even a large one) is not sufficient to infer intent – the effect has to be, in addition, objectively unlikely.

The most straightforward technique to automatically generate districting plans is to employ partition-generating functions to enumerate all of the possible districts that could be created from a given set of population units. Then, to obtain the set of all legal districting plans, the plans that do not meet legal criteria for acceptance are eliminated, such as those plans containing non-contiguous districts. For both practical and legal reasons, contiguity is a necessary constraint (we show in Appendix A that sampling random, non-contiguous districts yields results that are practically identical to at-large elections.)

A variant technique incorporates the contiguity constraint into the enumeration algorithm, so that only contiguous districts are generated (Shepherd and Jenkins 1970; Gudgin and Taylor 1979). This technique and its variants are inherently limited to very small problems, although its use continues to be

[7] Cirincione (et. al) apply their method to detect racial gerrymanders, but there is nothing in their general approach that limits its applicability. Indeed, precursors of their method were used to evaluate partisan intent (see, e.g., Rossiter and Johnston 1984).

more generally advocated for analytical purposes (McDonald and Engstrom 1991). As the number of population units to be incorporated into a plan increases, the number of possible districting plans grows exponentially and pure enumeration become practically impossible. When these techniques were first proposed, plans having only twenty-six population units to be assigned to districts were considered to be at the limits of tractability (Rossiter and Johnston 1981). Even a scenario with as few as fifty population units results in over 10^{31} potential districts, which is an enumeration problem well beyond the limits of current supercomputers. Full enumeration will remain beyond reach forever, if what most computer scientists believe about the theoretical limits of computation is correct (Altman 1997).

The intractability inherent in the enumerative approach to even any modest sized redistricting problem has led many social scientists to turn to *ad-hoc* methods for generating plans, referring to these as "random samples" of contiguous districting plans (Engstrom and Wildgen 1977; Rossiter and Johnston 1981; O'Loughlin 1984; Rogerson and Yang 1999; Cirincione, Darling, and O'Rourke 2000). However, the claim that these procedures can randomly sample is provably incorrect. We show in Appendix B that true random-sampling is computationally intractable in general for contiguous districts, in Appendix C provide a simple example showing that CDR's method (the only method proposed explicitly to test *predominant* intent) is statistically biased.

A general lesson to be drawn from the Appendices is that creating an algorithm that does "random things" is trivial, but it is dangerous to assume that such arbitrary behavior produces statistically random results. Unsurprisingly, we are not the first scholars to reach this conclusion. One of the most prominent computer science textbooks (Knuth 1997) warns against this assumption when designing random number generators. Another textbook cautions quite specifically, "Generating random permutations (and other combinatoric objects) is a an important little problem that people stumble upon and often botch up…you must be very careful with random (combinatorial) generation. We recommend that you try reasonably extensive experiments with any random generator before believing it" (Skiena 1988: 248).

The operations research community already generally recognizes that the solutions produced by automated redistricting algorithms are heuristic. Moreover our lengthy search of the computer science literature revealed a single method for producing partitions of sets with a known (in this case, uniform) random sampling distribution. These methods were invented over twenty years ago (Nijenhuis and Wilf 1978) but are not well known. Unfortunately, as the proof in Appendix B shows, it is unlikely that *any* algorithm can be developed to efficiently generate random sample from the set of all contiguous (or equal population, compact, etc.) redistricting plans for any even moderately large number of population units.

It is incumbent upon any author when developing a new method of estimation or modifying an existing one to demonstrate that their particular method is well behaved. Scholars cannot accept these other algorithms as legitimate methods of random sampling without evidence that they work correctly, especially when it can be shown that a closely related algorithm is biased.[8] None of the proponents of heuristic methods (in particular, Rossiter and Johnston 1981; Rogerson and Yang 1999; Cirincione, Darling, and O'Rourke, 2000) give any evidence to show that these methods produce true random samples. Instead, proponents of “computationally intensive” sampling choose to assume that the method of their invention produces results to their liking.

Section 4. A Sound Methodology for Assessing Predominant Intent Claims

[8] CDR’s argument that multiple algorithms produce "stratified" random samples (pg. 8) are unconvincing. The various algorithms that add to the contiguity requirements are unlikely to produce distributions of plans that over-sample from the population of contiguous districts, or from the distribution produced by their algorithm over that population, in any easily characterizable way.

Even if the “stratified” samples had been the results of over-sampling a known population, it would still be unclear what over-sampling means in the context of a continuous property like compactness. The authors claim to draw a “stratified” sample of compact plans - but what does this mean? How compact are the plans in the sample? Are they as compact as possible? Do they meet some threshold of “reasonable” compactness? All that we can safely assume, from what the authors tell us, is that these plans will probably be somewhat more compact than those drawn by their contiguity-only algorithm. This falls far short of what we would want to know before we started to make inferences about the properties of “compact” plans.

Regardless of the details of the statistical model, evaluating intent inevitably involves comparing counterfactuals. For example, in evaluating whether a plan was intended as a partisan gerrymander, we might ask, "What other goals could the legislature have plausibly intended?" If the motive were an incumbent gerrymander, how likely is it that the redistricting plan would resemble the current plan? If the motive were a partisan gerrymander, then how likely is it that the redistricting plan would resemble the current plan? Only if the plan was a likely result of one goal (or set of goals), and not of another, can we say that there is evidence that the legislature intended that particular goal.

Kousser (1991) describes the problem of assessing intent particularly well:

> When historians attempt to explain some event, they implicitly or explicitly choose between two or more possible explanations on the basis of the extant evidence, relevant theory, and analogies. To say that racial or sex discrimination motivated an action is to say that discrimination caused the action in some sense and that other possible rationales did not cause it, or were less important, or at least do not wholly exclude invidious discrimination as the cause. Explanation cannot be assessed independently, but only in relation other explanation" (Kousser 1991, 714).

Stated more formally, the probability that a redistricting plan is not a racial gerrymander given its characteristics can be expressed as a Bayes Factor, by dividing the probabilities of the observed or estimated characteristics (such as bias or responsiveness) conditional on a racial motivation by the probability of the observed characteristic under other hypotheses:

$$\frac{prob\left(y \in \{any\ other\ \ motive\} \mid \left\{\mathbf{C}, EV(\tilde{\mathbf{E}})\right\}\right)}{prob\left(y = racial\ gerrymander \mid \left\{\mathbf{C}, EV(\tilde{\mathbf{E}})\right\}\right)} =$$

$$\frac{prob\left(\left\{\mathbf{C}, EV(\tilde{\mathbf{E}})\right\} \mid y \in \{any\ other\ \ motive\}\right) \times prob(y \in \{any\ other\ motive\})}{prob\left(\left\{\mathbf{C}, EV(\tilde{\mathbf{E}})\right\} \mid y = racial\ gerrymander\right) \times prob(y = racial\ gerrymander)} << 1$$

In words, the legislature chooses some plan $\mathbf{p}^*$ from the set of all feasible plans, acting on some intent *y*. We want to infer the likelihood of the intent to create a racial gerrymander from observed characteristics,

$\mathbf{C}$, of the districts chosen and of the elections held within those districts, $\tilde{\mathbf{E}}$. A predictive model is usually applied to generate an expected value of election outcomes, $EV(\tilde{\mathbf{E}})$, rather than inferring long-range electoral outcomes from a single election. To calculate the likelihood of intent of a racial gerrymander, based on our observations, we need to know *both* the distribution of outcomes when intent is to create a racial gerrymander and the distribution of outcomes when the intent is otherwise (see Altman 2002 for a more detailed treatment of this approach).

Scholars who propose a sampling approach to determine intent attempt to estimate the distribution of $\{\mathbf{C}, EV(\tilde{\mathbf{E}})\}$ by drawing plans from a random sample, and claim a racial intent whenever the observed $\{\mathbf{C}, EV(\tilde{\mathbf{E}})\}$ of a given plan is in the far tails of the estimated distribution. Since, as we argued in the previous section, true random sampling of redistricting plans is intractable and all current methods are biased, the estimated distribution of $\{\mathbf{C}, EV(\tilde{\mathbf{E}})\}$ and any subsequent inferences based on that distribution are similarly biased.

Moreover, even an unbiased sample would not provide sufficient information to evaluate competing claims about intent, since no existing method claims to sample from the set of gerrymanders conditioned on other plausible intents. For example, if one *could* determine that the 99% of feasible redistricting plans had only one majority-minority district, and an adopted plan had three, one might well conclude that some intent to draw such districts was present, but one would still have no statistical basis for deciding between, for example, partisan versus racial predominant intent as the most likely explanation. More formally, the distribution of $\{\mathbf{C}, EV(\tilde{\mathbf{E}})\}$ derived from plans chosen at random is: $H_o : \{\mathbf{C}, EV(\tilde{\mathbf{E}})\} \mid y = no\, manipulation$. This distribution is sufficient to reject the null hypothesis that the districts were not manipulated, but not sufficient to calculate the likelihood of any particular motivation yielding that plan.

(Finally, even if one could sample from the set described above, to personal intent would require assuming that the distribution of available plans was also reasonably discoverable by the districter. This is implausible for past redistricting cases, although the discovery of an efficient sampling method might make this a more plausible assumption for future cases.)

This criticism applies to the ad-hoc methods we have seen, as well. For example, applying a standard *t*-test, as Viscuzi does, is not justified unless the establishing that the distribution of population deviations among neutral redistricting plans is a normal distribution. We are aware of no formal or empirical reason for this to be true in general, and no reason is given. Similarly, the 'boundary segment analysis' as described by the Court, implicitly relies on assumptions about the statistical distribution of boundary segments conditioned on the intent to gerrymander.[9] This distribution is likely to be impossible to compute correctly, for the reasons already discussed.

The erroneous application of statistical methods in the search for intent is more common in establishing claims of partisan gerrymandering. Practitioners and academics often use a simple re-aggregation of statewide or presidential elections, or voter partisan registration where such records are kept, to assess the likely partisan effects of a proposed redistricting plan. A more sophisticated method proposed by Gelman and King (1994) predicts expected votes within districts of alternative redistricting plans to estimate the shape of the seats-vote response curve, and associated measures of bias and responsiveness. This methodology has been used to aid in practitioners in redistricting and accepted in courts as evidence of intent to partisan gerrymander.[10]

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9 The 'boundary segment analysis' method is not publicly documented, thus we rely on the Court's description of the method.

10 The existence of 'bias' is neither necessary nor sufficient for partisan manipulation to have occurred. without a reference baseline to the total normal vote, i.e., the overall expected vote for a party within a given jurisdiction. For example, if the expected turnout in a state is 51% Democrat, someone seeking the maximum number of seats for the Democrats might propose a plan with zero 'bias', but a high responsiveness (Kousser 1996). Conversely, a 'biased' plan could have result from a planner, blind to partisan consequences, who wishes to maximize the number of competitive districts. Thus a "fair" plan might be one that satisfies many criteria simultaneously (Niemi & Deegan 1978).

But, while this method is quite useful for predicting partisan effect, which may in turn suggest intent, it cannot by itself show intent. Without knowledge of the expected partisan consequences of reasonably available *alternative* redistricting plans, no inference can properly be made regarding partisan intent, even though expert witnesses often make such inferences.

A Method of Revealed Preferences

In lieu of directly calculating the likelihood that the observed plan was generated by a particular motive, we propose that intent can be probed by examining reasonable plans that were *not* chosen. Whereas sampling and more simplistic approaches attempt to make inferences by sampling from a **global** distribution of plans to calculate the probability of the current plan under the **null** hypothesis, we take the opposite approach: we use information from **the local distribution of plans "near" the actual choice** to eliminate competing **alternative hypotheses** of intent.

Courts and litigants use revealed preferences informally, especially in the absence of smoking gun evidence, when they examine characteristics of plans that were rejected to illuminate why a particular plan was accepted. It has also been used by academics to assess intent in North Carolina's redistricting in the 1990s (Gronke and Wilson 1999). The method can establish a group's most preferred plan, but a full rank ordering is not possible, and suggests an alternative method of assessing intent based on revealed preferences. We introduce two innovations to this methodology. The first is to formalize this method. The second is by showing how computationally-intensive methods could be modified or human drawn maps applied to provide evidence of predominant intent.

A fundamental axiom in economics is the Weak Axiom of Revealed Preference (WARP) (Samuelson 1948, also see Varian 2002 for a modern introduction). Any method that is used to infer preferences from the actions of a rational actor must rest on WARP. WARP states that if one observes a choice of $\{a\}$ from a set $\{a,b,c\}$ than it must be the case that $a \geq b, a \geq c$. WARP simply states that if I like chocolate ice cream over vanilla and strawberry, I will choose chocolate when presented with either a choice between chocolate and vanilla or chocolate and strawberry. In a redistricting context, if plans *a* and b are available, but plan a is chosen, than it must be that plan a is weakly preferred. Using this method to reject competing hypotheses does *not* require the distributional assumptions that limit sampling heuristics. WARP is deterministic: the probability that $b > a$ when a is chosen, equals zero.[11]

For example, assume that plans a and b are identical in all respects that are legally and politically relevant to the redistricter *except* that plan b yields an additional minority-majority district. If plan a is chosen, we can infer that the redistricter did not intend to maximize minority-majority districts, in other words that minority-majority district maximization was not the predominant (overriding) intent.

More generally, if plans $\{a_1,...,a_n\}$ are identical in all important respects, except for the number of expected minority-majority and partisan districts in each plan, we can map the boundaries of the redistricting planner's willingness to trade minority-majority seats for partisan seats from the observed plans.

In theory, WARP could be used to make inferences about preferences using the alternate plans proposed in the public record. A caution is that one cannot clearly infer the preferences when alternative proposals differ in multiple significant ways. For example, if plan a yields {11 Democratic seats, 3 minority-majority seats, and 15 safe seats for incumbents}, while plan b yields {10 Democratic seats, 4 majority-minority seats, and 16 safe seats for incumbents} then one can not determine whether partisanship, incumbency, or race "predominated" in the choice.

There are two approaches by which this method can be applied in courtrooms. First, judges can evaluate plans that were introduced during the redistricting process to reveal intent. For example, in *Minority Coalition* v *Arizona Independent Redistricting Commission* CV 2002-004380 state Judge Fields found that a map rejected by the Arizona Independent Redistricting Commission with more competitive

[11] Any method for inferring intent must assume some weak collective rationality – that what is intended is also what is chosen. If this assumption is violated, and the preferred plan is not the one chosen (at least probabilistically), then any attempt to infer intent is futile.

districts than the commission's adopted map served as evidence that the commission did not adhere to the state's constitutional mandate to draw competitive districts to the extent practicable. Second, litigants can produce plans of their own that could have reasonably been drawn by the redistricting authority. In a case study that we describe in more detail, in *Rodriguez* v *Pataki* No. 02 Civ. 0618, Plaintiffs devised a minority-maximization representation plan to demonstrate that their own proposed map was not a minority maximization map.

Heuristic Exploration of Alternatives

Computationally intensive district generation techniques, such as those used by CDR and others, may be applied to reveal intent using WARP, with two extensions. The first extension is to use these techniques to draw plans that attempt to optimize on some value function that incorporates all relevant political goals. The second extension is to use a given plan under evaluation or the same "starting values" that most nearly generated the given plan, as a starting point. More fully, the steps for generating plans are as follows:

1. Divide the relevant characteristics of a plan into three categories.
    a. Criteria that describe the feasible set of districts, **K**, such as the maximum population deviation between districts, contiguity, etc.
    b. A characteristic, **I**, that best proxies the intent you wish to test.
    c. Characteristics, **R**, representing any other politically relevant criteria.
2. Enter the current plan, and any alternative plan that is part of the public record, into a GIS system, along with the data necessary to evaluate **K**, **I**, and **R**.
3. Use these plans as starting points for any heuristic local optimization algorithm, such as simulated annealing, genetic algorithms, or the greedy heuristic described in CDR.
4. Use the optimization algorithm to search for a plan $p*$ that such that:

$$\mathbf{I}(p^*) > \mathbf{I}(p), \quad p^* \in \mathbf{K}$$

If a feasible p^* exists, then the motive proxied by **I** cannot strictly have been overriding (lexically preferred).

5. To explore the trade-off among criteria in a more nuanced way - use the optimization algorithm to search for a plan p^{**} that such that:

$$\mathbf{I}\left(p^{**}\right) > \mathbf{I}(p), \quad \mathbf{R}\left(p^{**}\right) = \mathbf{R}(p), \quad p^{**} \in \mathbf{K}$$

And then maximize **I**, subject to holding all but one (j) of the other relevant criteria constant:

$$\max_{p^* \in \mathbf{K}, \left\{R_i\left(p^*\right) = R_i(p) | i \neq j\right\}} \mathbf{I}\left(p^*\right)$$

Thus revealing boundaries on the willingness to increase **I**, or to trade some of **I** for any of the other criteria.

Applying the technique is more straightforward than the proceeding formalism suggests. For example, let **MIN** be the number of minority-majority districts in a particular plan and let **DEM** and **REP** be the number of expected Democratic and Republican districts. If we can find a legal (contiguity, equal populous, etc.) modification of the current plan, p^* that increases **MIN** substantially without changing **DEM** or **REP,** we have good reason to conclude that racial intent did not predominate. In this case, **I** = {number of minority-majority seats}, **K =** {contiguity, legal population deviation}, and **R** = {number or Republican seats, number of Democratic seats}.

Restrictions and Ramifications

The heuristic approach described above, while within the capabilities of well-known optimization algorithms, when expertly applied, is beyond the capabilities of currently available commercial redistricting software.[12] Until powerful automated redistricting software becomes more readily available,

12 Browdy (1991) notes that simulated annealing could be used for such multi-criteria optimization. Moreover, the Texas Automated Redistricting Plan Generation and Evaluation Tool (TARGET) software, developed by the research branch of the Texas Legislative Council is a notable exception. Although not commercially available, it is able to perform practical multi-criteria optimization of districting plans, as required by the heuristic we describe.

manual analysis of alternative plans will continue to be the most practical application of this methodology.

Whether automated or manual, this methodology has four important limitations. First, the choice of **K**, **I**, and **R** are often contextual. These must represent plausible explanations of the goals of the redistricting process in a given jurisdiction. For example, while some characteristics such as equal population are constitutionally required for all states, others, such as minimizing county splits, depend on state law. Like any other statistical test, a set of reasonable causal hypotheses must be a starting point.

Second, depending on the set of plans revealed by the heuristic exploration, this method might fail to reject any of the competing hypotheses. This is analogous to statistical tests where data may not contain enough information to provide useful estimates. A common solution is to "obtain more data," which in this context means finding other plans that were under consideration, for example, by seeking other plans in the public record that can be used as starting points for improvement on a criterion of interest.

Third, the plans used for comparison must have been reasonably discoverable by the redistricting planner. The redistricters will not have examined every alternative, which for even a modest number of population units are practically infinite. However, if the planner could easily have discovered an alternative, had they preferred it, the method still applies: For example, if a redistricter did not even bother to search for a plan that added another majority-minority seat and that plan was easily discoverable, we can still reasonably infer that the planner preferred the chosen plan over the one with more minority-majority seats. Note that this discoverability assumption is much more plausible using our method than for statistical methods: Our method requires that the districter be aware only of plans that were directly related to proposed plans, while the statistical methods assume that the districter was aware of the entire distribution of plans.

Section 5. An Illustration: Redistricting the New York State Senate

Although we formalize the concept of revealed preferences in the redistricting context, practitioners already use such methods in an informal way. The case *Rodriguez* v *Pataki* No. 02 Civ.

0618, which concerned the 2001 redistricting of New York's state Senate districts, illustrates the use of this technique. In 2001, New York had a divided state legislature.

Faced with the prospect of a deadlock in the redistricting process, the legislative leadership of the Democratic House and Republican Senate compromised by drawing districts for their respective chambers (McDonald 2004). The Senate plan protected Republican interests in the state by three methods. First, districts in and around New York City were purposely overpopulated within the 10% deviation allowed under *Karcher* v *Daggett* 462 U.S. 725 (1983) to squeeze an additional Republican Senate district into the underpopulated districts of upstate New York. Second, the minority population of Suffolk and Nassau Counties were diluted by cracking their population among several Republican dominated Senate districts. Third, a Republican district was drawn in the Bronx by over-stacking minority population in the surrounding districts.

To win a Section 2 claim of the Voting Rights Act under the *Gingles* test, Plaintiffs had to show that minority majority districts capable of electing minority candidates of choice could be drawn in Nassau and Suffolk counties. Plaintiffs proposed a plan with two minority-majority districts. Defense contended that these districts were a clear effort to maximize solely the number of minority districts. To counter this claim, Plaintiffs devised what they called a "reverse-maximization" plan. The plan reversed the population deviations in the adopted Senate map to underpopulate the districts in and around New York City and maximized the number of minority districts. They showed that they could have drawn another minority majority district. In our formal terms, Plaintiffs wished to show that a maximizing solely the number of minority-majority districts (**I**) within the constraints of contiguity and equal population (**C**) as found in the adopted plan (p), there was a plan which had more minority districts (p^*) than the Plaintiff's plan (p'), and hence the motive behind (p') was not *predominantly* racial.

Defense argued that the "reverse-maximization" plan (p^*) was not sufficiently compact. They argued further that this lack of compactness demonstrated that Plaintiff's plan (p') was motivated predominantly by race, since (p^*) could be drawn only by subverting traditional redistricting principals. Thus, ($p^* \notin \mathbf{K}$), i.e., that (p^*) was not among the legally acceptable redistricting plans.

The Defense argument was incorrect within the context of revealed preference theory. If Defense was correct that (p^*) was not a legal districting plan, then nothing could be said about the tradeoff between (p^*) and (p') to illuminate if race predominated (p'). Furthermore, New York does not have a compactness requirement, and thus ($p^* \in \mathbf{K}$); it is a legal redistricting plan. Compactness only serves as an indicator of predominant racial intent *vis a vis* the U.S. Supreme Court. That the "reverse-maximization" plan (p^*) intended to maximize race, and it subverted compactness to do so, only reinforces that race predominantly motivated (p^*).

The Federal court upheld Plaintiff's argument that race did not predominate in their proposed plan (p'). Nevertheless, in its final decision the court rejected the Plaintiffs for failing the *Gingles* test, finding that the proposed minority districts were not sufficiently likely to elect a minority candidate of choice.

Conclusion

Much of our discussion regarding using effect to determine redistricting intent should come as no surprise to those familiar with classical hypothesis testing. A hypothesis test is framed to test a null hypothesis against an alternative. While a null hypothesis can be rejected, the alternative theory that it is tested against cannot be proved. There are always other, untested, alternative explanations that may more fully explain the observed phenomenon under examination.

Statistical tests for intent in redistricting often fail to appreciate this point. Even if we knew the true distribution of characteristics of redistricting plans (which is generally computationally intractable) and even if we can meaningfully assign collective intent to a legislative body or redistricting commission (which, in at least some circumstances, we cannot), the proposed tests of redistricting intent are usually constructed against a null hypothesis of "no intent" rather than showing which of *all* plausible theories of intent is the most likely explanation. By accepting these tests as evidence, courts risk the danger of improperly assigning intent, for example, to maximizing the number of minority-majority districts when other untested and equally plausible motivations, such as maximizing partisan outcomes or protection of incumbents, may have driven the creation and adoption of a redistricting plan.

To address this shortcoming, we propose a method of revealed preferences to deduce intent. We use the weak axiom of revealed preference to turn hypothesis testing on its head and ask the question what did redistricters *avoid*? What they chose *not* to do can reveal their intentions. This approach is used informally in courts; our contribution is to formalize the method and suggest how computationally intensive methods may be applied in this context.

Justice Kennedy's concurrence in *Vieth* v. *Jubelirer* (2004) optimistically opines "…new technologies may produce new methods of analysis that make more evident the precise nature of the burdens gerrymanders impose on the representational rights of voters and parties" Id., at 8. We, too, are optimistic that advances in statistics and technology may result in new and unforeseen methods to evaluate redistricting intent through results. Beyond offering our own method, by formalizing an economics-based method to evaluate redistricting intent, we hope to guide future statistical methods.

Appendix A: Randomly Drawn, Non-Contiguous Districts are Equivalent to at-Large Elections

Here we show that randomly drawn *non-contiguous* districts yield results that are roughly equivalent to at large elections. A formal asymptotic proof is unnecessary, since a Monte Carlo simulation shows that for any practical purpose, random districts will yield the same outcome as at-large elections.

Suppose that a state has a population of *N* people, each of whom is randomly assigned into one of *K* equally-populated districts.[13] Suppose also that 51% of the total population are "Democrats" and 49% of the total population are "Republicans", that turnout is 100%, and that persons vote only for a candidate of their party. This scenario provides a compelling test case: if random-districting differed from at-large elections at all, we would be most likely to see these differences when the margin between the parties was very slim.

In an at-large election, a Democrat would win every seat. What percentage of the time do they win every seat in random districts? Using Monte Carlo simulation (see Fishman 1996), we can see that in moderately-sized districts, the Democrats are virtually guaranteed to sweep the elections, despite their narrow margin:

Seats *District Population Size (# of people in a district)*	*5*	*10*	*20*
1001	14%	0.8%	0%
5001	73%	45%	16%
10001	94%	83%	65%
50001	**100%**	**100%**	**100%**

Probability that Democrats Win All Random Districts

13 In formal terms, we are partitioning the population by sampling *without* replacement. The alternative sampling with replacement, implies essentially that every voter randomly votes for a Democrat or Republican with some fixed independent probability. As a comparison, we also ran simulations *with* replacement (using a Binomial distribution). For sufficiently large districts, the results are identical to the monte-carlo estimate presented here. And although the two methods differ mathematically for small districts, the substantive implications are the same.

Appendix B: Random Sampling of Contiguous, Equal Population Districts is Computationally Intractable

In this appendix, we show that any algorithm that could produce true random samples of redistricting plans is extremely likely to be computationally intractable when applied to redistricting problems of significant size, i.e., even a modest number (50) of census blocks to be assigned to districts. In formal terms, we show that the sampling problem itself is "NP-complete." NP-complete problems are a special class of partitioning problems that computer scientists generally consider are computationally intractable (Papadimitriou 1994). Although it still remains possible that for particular cases of an NP-hared problem, contiguity or other constraints will reduce the number of possible partitions to a tractable size. This is an extension of a previous scholar's work (Altman 1997), but in the interests of precision, we repeat portions of proofs here.

Notation:

- $\mathbf{x}_i$ refers to the i^{th} census block. These blocks are vector-valued, and we will assume that blocks comprise different population values, partisan registration percentages, and geographic locations, among other features.
- $\mathbf{d}_i$ refers to the i^{th} district. A district is a set of census blocks: $\mathbf{d}_i = \{\mathbf{x}_j, \mathbf{x}_k, \ldots, \mathbf{x}_n\}$.
- $\mathbf{p}_i$ refers to a particular plan. A plan is a partition of the set of all census blocks into a set of districts: $\mathbf{p}_i = \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$.

Definition:

The population inequality of a plan, $\mathrm{s}(p_i)$, is defined as the difference between the population of the largest and the smallest districts:

$$\mathrm{s}(p_i) = \max[\mathrm{pop}(d_i)] - \min[\mathrm{pop}(d_i)], d_i \in p.$$
$$\text{Where, } \mathrm{pop}(d_i) = \sum\nolimits_{x_i \in d_i} \mathrm{pop}(x_i),\ \ x_i \geq 0.$$

Proof:

A common method to show that a problem is NP-complete is to show that it can be reduced to a *reference problem* that is known to be NP-complete. Suppose algorithm **A** produces a random sample of ***N*** contiguous plans, with population deviation smaller than a specified threshold *K*. Let *N=1:* clearly **A** generates a single contiguous, equal-population plan, **p***. Formally, using the notation above, **p*** is a partition where edges fully contained in each district form a connected graph, and *K*> s(***p***). Given $\mathbf{p}^*$ it is then trivial to decide the question (for some given constant *K* and increasing *N*): "Is there a plan such that all districts are contiguous and s($\mathbf{p}^*$)<*K*?" Answering the question is equivalent to solving the following problem, which is strongly NP-complete (Johnson 1982, following his notation, referring to work by Frieze, later published as Dyer & Frieze 1985):

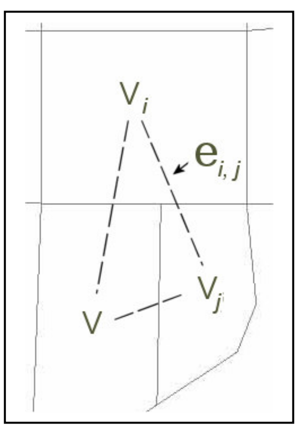
Cut into Connected Components of Bounded Weight

Instance:[14] Graph G = (V,*E*), a size $s(v) \in Z^+$ for each *a*,

Problem: Is there a partition of vertices, **V,** into disjoint sets $\mathbf{V}_1$ and $\mathbf{V}_2$ such that

$\sum_{v \in \mathbf{V}_1} s(v) \le K$ and $\sum_{v \in \mathbf{V}_1} s(v) \le K$ and both $\mathbf{V}_1$ and $\mathbf{V}_2$ induce connected subgraphs of **G***?*

This reference problem maps directly to the redistricting of a jurisdiction into two districts. In redistricting terms each vertex $\mathbf{v}_i$ is a census block, and each edge $\mathbf{e}_{i,j}$ denotes physical adjacency of those blocks. Figure 1 provides a graphical illustration:



[14]Note that the minimum sum of squares problem assumes that each set member is positively valued. This makes it a subset of the competitive redistricting problem, because Democrats may outnumber Republicans in a census block. Hence this demonstration shows that the redistrict is NP-hard but is not sufficient to show that it is NP-complete.

Figure 1. Notation for Census Blocks

The weight of each vertex, s(v), is simply the population in each census block. So, the sum of the values of s(v) must be below a threshold K, or in redistricting terms the population deviation between districts.

Technically, this proof applies only to deterministic optimization algorithms. However, while it is still theoretically possible that an efficient stochastic or approximate method exists, related partitioning problems have been shown to be unapproximable (Zuckerman 1996) and it is widely believed by computer theorists that no stochastic methods are efficient for NP-complete problems. (Kabanets 2002)

Appendix C: An Example Showing that the CDR Method is Biased

A simple example suffices to prove the basic "computationally-intensive" method used by CDR for drawing contiguous districts does not produce a representative sample of the population of districting plans. Consider a simple "state" which is composed of six identically sized and populated block groups on a 2x3 grid. Each block-group is contiguous to its horizontal and vertical neighbors.[15] We divide the state into two districts, each containing exactly three blocks and we fully enumerate the possible districting plans and calculate the probability the CDR algorithm will find each solution. The algorithm is biased if all solutions do not have the same probability of discovery.

In this hypothetical state, there are three possible districting plans (assuming that the numbering of districts is unimportant, but this assumption does not fundamentally affect our conclusions), as shown below:

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[15] Note, that by constructing the geography with 'holes' so that no population units meet at a single point we avoid the issue of whether to treat such units as contiguous. This simplification is only taken to clarify the exposition. In fact, decisions about how to measure contiguity (and other criteria) are completely independent from the sampling behavior of the algorithm. Our example does not rely on the shapes of the population units being as depicted in the drawing -- it relies only that the implied set of contiguity relationships not be impossible, *a priori*.

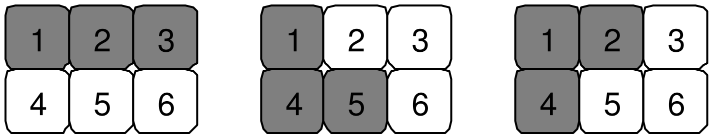

Figure 1: feasible contiguous redistricting plans on a 2x3 grid.

The contiguous district generation algorithm used by CDR is very simple, and thus amenable to formal analysis. As they describe it (196):

> The first algorithm, the *contiguity algorithm*, begins by randomly selecting a block group to serve as the "base" of the first district. It then constructs a "perimeter list" containing the unassigned block groups contiguous to the base block group. The program then randomly selects a block group from the perimeter list to add to the emergin district and adjusts the perimeter lists. The process continues until the population of the emerging district achieves the desired population level. …The next district begins with the random selection of a census block group from among those that touch one of the complete districts. [And the process continues until a legal plan is generated, or until no more legal districts can be created, in which case the process is restarted.]

In a true random sample of contiguous districting plans, the probability of the method generating each plan should be 1/3. Using the computationally intensive "sampling" method, the probability is lower for {1,2,3} than for the other two plans.[16] What is the probability of generating each districting plan using CDR's algorithm? The tree below shows all of the possible sequences of choices starting from the base block groups 1,2,3 (the paths from the bases 4,5,6 are symmetric):

---

[16] At first glance, one might think this bias is intended to select "compact" districts. In fact, this method is supposed, by Cirincione et. al, to select from the universe of contiguous plans. They use a different algorithm to select compact plans.

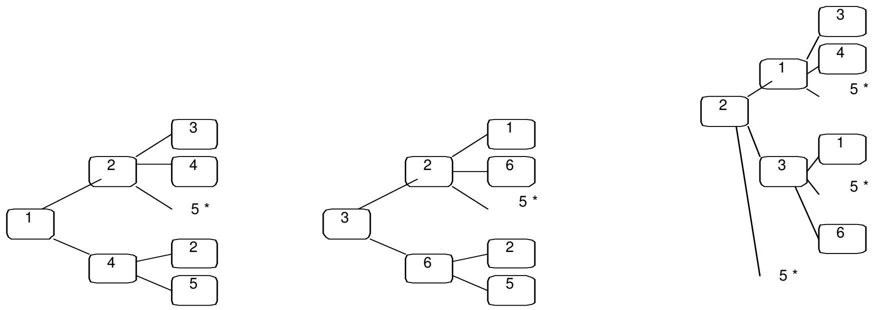

Figure 2: Event-trees showing the generation of the district plans in Figure 2. Each sub-tree is equally likely (P=1/6), and the probability of following any branch at each node is equal to 1/(number of branchs). Starred nodes indicate illegal plans, which cause the algorithm to restart with subtree selection.

The total probability of generating plan {1,2,3} during a single run of CDR's algorithm is the probability of the sum of the probabilities of the paths ({1,2,3}, {3,2,1}, {2,1,3}, {2,3,1}}, multiplied by two (for symmetry with starting points (4,5,6)). This equals (1/36+1/36+1/54+1/54)*2=5/27. The probability of generating plan {1,4,5} (which is the same as the probability of {1,2,4} by symmetry) is the sum of the probabilities of the paths {{1,2,4},{1,4,2},{2,1,4},{3,5,6}}*2, which equals (1/36+1/24+1/54+1/24)*2=7/27. (The probability of having to start over is 8/27, but this does not affect the asymmetry of accepting each plan in later rounds.).

Thus, the basic algorithm used by CDR is statistically biased. Admittedly, this is a small-sample result, and the asymptotic bias of the algorithm may be different than its small sample bias. Note, however, that unlike sampling techniques based on statistical theory, *the heuristic techniques used by CDR and others carry with them no guarantees of unbiasedness* or any other asymptotic properties. They cannot appeal to the large of law numbers, which is often used in proofs of more traditional statistical models. In other words, no one has yet supplied a theoretical or empirical reason to justify the belief that the bias we reveal here will fix itself in larger districting problems.

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